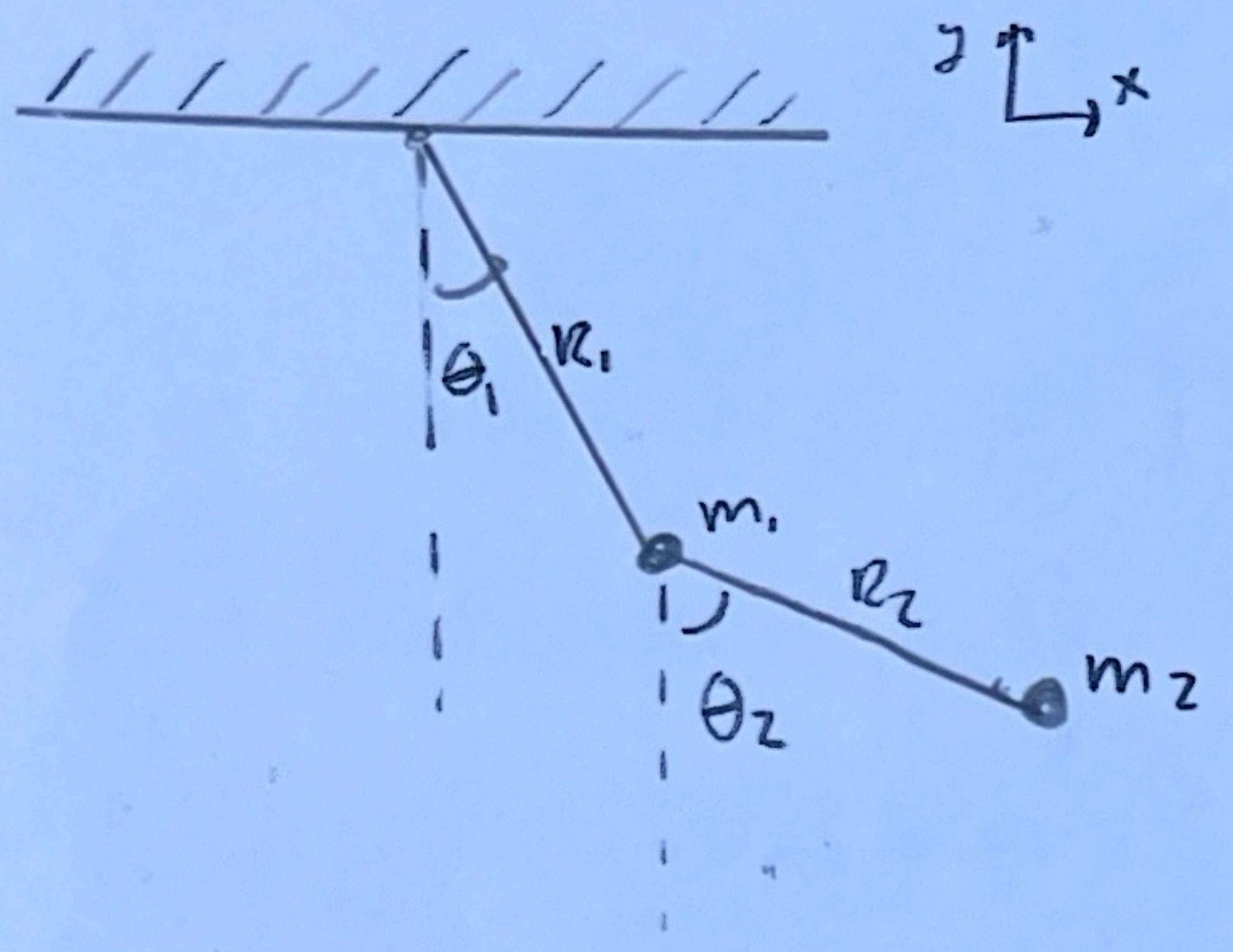
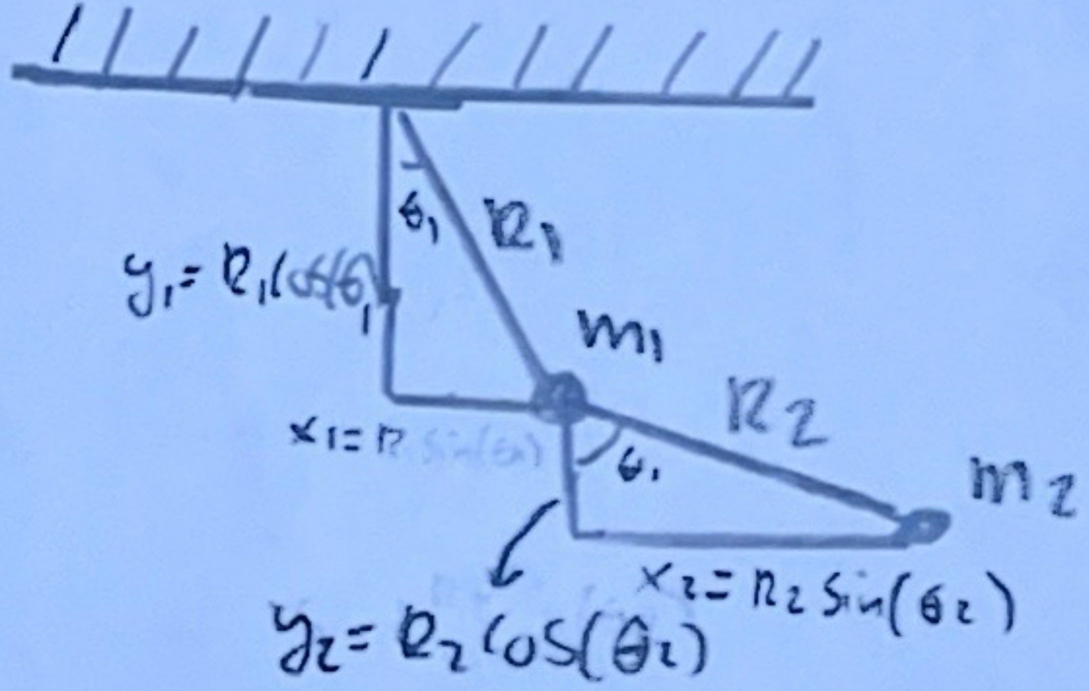
Double Pendulum



$$L = T - U$$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{dL}{d\dot{q}_i} = 0$$

$$\dot{q}_i = \frac{dq_i}{dt} \; , \; \ddot{q}_i = \frac{d^2 q_i}{dt^2}$$



$$T = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2$$

$$U = m_1 g h_1 + m_2 g h_2$$

$$r_1 = \begin{pmatrix} R_1\sin(\theta_1) \\ -R_1\cos(\theta_1) \\ 0 \end{pmatrix} \qquad r_2 = \begin{pmatrix} R_1\sin(\theta_1) + R_2\sin(\theta_2) \\ -R_1\cos(\theta_1) - R_2\cos(\theta_2) \\ 0 \end{pmatrix}$$

not constant

$$\dot{r}_1 = \begin{pmatrix} R_1\cos(\theta_1)\cdot\dot{\theta}_1 \\ +R_1\sin(\theta_1)\cdot\dot{\theta}_1 \\ 0 \end{pmatrix} \qquad \dot{r}_2 = \begin{pmatrix} R_1\cos(\theta_1)\dot{\theta}_1 + R_2\cos(\theta_2)\cdot\dot{\theta}_2 \\ +R_1\sin(\theta_1)\dot{\theta}_1 + R_2\sin(\theta_2)\cdot\dot{\theta}_2 \\ 0 \end{pmatrix}$$

$$\dot{r}_1^2 = \dot{r}_1 \cdot \dot{r}_1^T = \begin{pmatrix} R_1\cos(\theta_1)\cdot\dot{\theta}_1 \\ R_1\sin(\theta_1)\cdot\dot{\theta}_1 \\ 0 \end{pmatrix} \begin{pmatrix} R_1\cos(\theta_1)\cdot\dot{\theta}_1 & R_1\sin(\theta_1)\cdot\dot{\theta}_1 & 0 \end{pmatrix}$$

$$= (R_1\cos(\theta_1)\cdot\dot{\theta}_1)^2 + (R_1\sin(\theta)\dot{\theta}_1)^2 + 0^2$$

$$= R_1^2\cos^2(\theta)\cdot\dot{\theta}_1^2 + R_1^2\sin^2(\theta)\dot{\theta}_1^2$$

$$= R_1^2\dot{\theta}_1^2(\cos^2(\theta) + \sin^2(\theta))$$

$$= R_1^2\dot{\theta}_1^2$$

$$\boxed{T_1 = \frac{1}{2} m_1 \cdot R_1^2 \dot{\theta}_1^2} \; , \; \boxed{U_1 = -m_1 g \cdot R_1\cos(\theta)}$$

$$\dot{r}_2^2 = \dot{v}_2 \cdot \dot{v}_2^T = \begin{pmatrix} R_1\dot{\theta}_1\cos(\theta_1) + R_2\dot{\theta}_2\cos(\theta_2) \\ R_1\dot{\theta}_1\sin(\theta_1) + R_2\dot{\theta}_2\sin(\theta_2) \\ 0 \end{pmatrix} \left(R_1\dot{\theta}_1\cos(\theta_1) + R_2\dot{\theta}\cos(\theta_2) \quad R_1\dot{\theta}\sin(\theta_1) + R_2\dot{\theta}_2\sin\right.$$

$$= [R_1\dot{\theta}_1\cos(\theta_1) + R_2\dot{\theta}_2\cos(\theta_2)]^2 + [R_1\dot{\theta}_1\sin(\theta_1) + R_2\dot{\theta}_2\sin(\theta_2)]^2$$

$$= [R_1^2\dot{\theta}_1^2\cos^2(\theta_1) + R_2^2\dot{\theta}_2^2\cos^2(\theta_2) + 2R_1R_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1)\cos(\theta_2)] + \cdots$$

$$\cdots [R_1^2\dot{\theta}_1^2\sin^2(\theta_1) + R_2^2\dot{\theta}_2^2\sin(\theta_2) + 2R_1R_2\dot{\theta}_1 \cdot \dot{\theta}_2\sin(\theta_1)\sin(\theta_2)]$$

$$= [R_1^2\dot{\theta}_1^2\cos^2(\theta_1) + R_1^2\dot{\theta}_1^2\sin^2(\theta_1)] = R_1^2\dot{\theta}_1^2\overbrace{(\cos^2(\theta_1) + \sin(\theta_1))}^{=1}$$

$$= [R_2^2\dot{\theta}_2^2\cos^2(\theta) + R_2^2\dot{\theta}_2\sin(\theta_2)] = R_2^2\dot{\theta}_2^2(\cos^2(\theta_1) + \sin^2(\theta_1)$$

$$= (2R_1R_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1)\cos(\theta_2) + 2R_1R_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1)\sin(\theta_2)$$

$$= 2R_1R_2\dot{\theta}_1\dot{\theta}_2(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2))$$

trick:

$$\cos(\theta_2)\cos(\theta_1) + \sin(\theta_1)\sin(\theta_2) = \cos(\theta_1 - \theta_2)$$

$$= 2R_1R_2\dot{\theta}_1\dot{\theta}_2 \cdot \cos(\theta_1 - \theta_2)$$

$$\dot{r}_2^2 = R_1^2\dot{\theta}^2 + R_2\dot{\theta}_2^2 + 2R_1R_2\dot{\theta}_1\dot{\theta}_2 \cdot \cos(\theta_1 - \theta_2)$$

$$\boxed{T_2 = \frac{1}{2}m_2(R_1^2\dot{\theta} + R_2\dot{\theta}_2^2 + 2R_1R_2\dot{\theta}_1\dot{\theta}_2 \cdot \cos(\theta_1 - \theta_2))}$$

$$U_2 = -m_2 g(-R_1\cos(\theta_1) - R_2\cos(\theta_2))$$

$$\boxed{U_2 = m_2 g(R_1\cos(\theta_1) + R_2\cos(\theta_2))}$$

$$\boxed{T_1 = \frac{1}{2} m_1 R_1^2 \dot{\theta}_1^2}$$

$$\boxed{U_1 = -m_1 g R_1 \cos(\theta)}$$

$$\boxed{T_2 = \frac{1}{2} m_2 \left(R_1^2 \dot{\theta} + R_2 \dot{\theta}_2^2 + 2R_1 R_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right)}$$

$$\boxed{U_2 = m_2 g \left(R_1 \cos(\theta_1) + R_2 \cos(\theta_2)\right)}$$

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$$L = T - U = T_1 + T_2 - U_1 - U_2$$

$$L = \frac{1}{2} m_1 R_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(R_1^2 \dot{\theta} + R_2 \dot{\theta}_2 + 2R_1 R_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right) + m_1 g_1 R_1 \cos(\theta_1) \dots$$

$$+ m_2 g \left(R_1 \cos(\theta_1) + R_2 \cos(\theta_2)\right) \rightarrow \begin{cases} = m_1 g_1 R_1 \cos(\theta_1) + m_2 g R_1 \cos(\theta_1) + m_2 g R_2 \cos(\theta_2) \\ = g R_1 \cos(\theta_1)(m_1 + m_2) + m_2 g R_2 \cos(\theta_2) \end{cases}$$

$$L = \frac{1}{2} m_1 R_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(R_1^2 \dot{\theta}_1^2 + R_2 \dot{\theta}_2^2 + 2R_1 R_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right) + g_1 R_1 \cos(\theta_1)(m_1 + m_2) +$$
$$+ m_2 g R_2 \cos(\theta_2)$$

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} = 0$$

$$\frac{\partial L}{\partial \theta_1} = -\frac{1}{2} \cdot 2 m_2 R_1 R_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \cdot 1 - g R_1 \sin(\theta_1)(m_1 + m_2) \cdot 1$$
$$= -m_2 R_1 R_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - g R_1 \sin(\theta_1)(m_1 + m_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = \frac{1}{2} \cdot 2 m_1 R_1^2 \dot{\theta}_1 \cdot 1 + \frac{1}{2} \cdot 2 m_2 R_1^2 \dot{\theta}_1 \cdot 1 + \frac{1}{2} \cdot 2 m_2 R_1 R_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \cdot 1$$

$$= m_1 R_1 \dot{\theta}_1 + m_2 R_1^2 \dot{\theta}_1 + m_2 R_1 R_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} = m_1 R_1^2 \ddot{\theta}_1 + m_2 R_1^2 \ddot{\theta}_1 + m_2 R_1 R_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 R_1 R_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$

$$= R_1^2 \ddot{\theta}_1 (m_1 + m_2) + m_2 R_1 R_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 R_1 R_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} = -m_2 R_1 R_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - g R_1 \sin(\theta)(m_1 + m_2) \dots$$
$$-\left(R_1^2 \ddot{\theta}_1 (m_1 + m_2) + m_2 R_1 R_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 R_1 R_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\right)$$

$$\begin{cases} - m_2 R_1 R_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \\ + m_2 R_1 R_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) \end{cases}$$

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} = -g R_1 \sin(\theta_1)(m_1 + m_2) - R_1^2 \ddot{\theta}_1 (m_1 + m_2) -$$
$$m_2 R_1 R_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 R_1 R_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \theta_2} - \frac{d}{dt} \cdot \frac{\partial L}{\partial \dot{\theta}_2} = 0$$

$$L = \frac{1}{2} m_1 R_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (R_1^2 \dot{\theta}_1^2 + R_2 \dot{\theta}_2^2 + 2 R_1 R_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + g_1 R_1 \cos(\theta_1)(m_1 + m_2) \ldots$$
$$+ m_2 g r_2 \cos(\theta_2)$$

$$\frac{\partial L}{\partial \theta_2} = -\frac{1}{2} \cdot 2 m_2 R_1 R_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \cdot (-1) - m_2 g R_2 \sin(\theta_2) \cdot 1$$
$$= m_2 R_1 R_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g R_2 \sin(\theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{2} \cdot m_2 R_1^2 \dot{\theta}_2 \cdot 2 + \frac{1}{2} \cdot 2 \cdot m_2 \cdot R_1 R_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$
$$= m_2 R_1^2 \dot{\theta}_2 + m_2 R_1 R_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt} \cdot \frac{\partial L}{\partial \dot{\theta}_2} = m_2 R_1^2 \ddot{\theta}_2 + m_2 R_1 R_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 R_1 R_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$

$$\left[\frac{\partial L}{\partial \theta_2}\right] - \left[\frac{d}{dt} \cdot \frac{\partial L}{\partial \dot{\theta}_2}\right] = \left[m_2 R_1 R_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g R_2 \sin(\theta_2)\right] \ldots$$
$$- \left[m_2 R_1^2 \ddot{\theta}_2 + m_2 R_1 R_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \underbrace{m_2 R_1 R_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)}_{-m_2 R_1 R_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 R_1 R_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)}\right]$$

$$\frac{\partial L}{\partial \theta_2} - \frac{d}{dt} \cdot \frac{\partial L}{\partial \dot{\theta}_2} = -m_2 g R_2 \sin(\theta_2) - m_2 R_1^2 \ddot{\theta}_2 - m_2 R_1 R_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + m_2 R_1 R_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt} \cdot \frac{\partial L}{\partial \dot{\theta}_1} = -gR_1 \sin(\theta)(m_1+m_2) - R_1^2 \ddot{\theta}_2 (m_1+m_2) - m_2 R_1 R_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) - m_2 R_1 R_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2$$

$$\frac{\partial L}{\partial \theta_1} = \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\theta}_1}$$

$$-gR_1 \sin(\theta)(m_1+m_2) = R_1^2 \ddot{\theta}_1 (m_1+m_2) + m_2 R_1 R_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + m_2 R_1 R_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2$$

$$\frac{\partial L}{\partial \theta_2} - \frac{d}{dt} \cdot \frac{\partial L}{\partial \dot{\theta}_2} = -m_2 g R_2 \sin(\theta_2) - m_2 R_1^2 \ddot{\theta}_2 - m_2 R_1 R_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) + m_2 R_1 R_2 \dot{\theta}_1^2 \sin(\theta_1-\theta_2)$$

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_2}$$

$$-m_2 g R_2 \sin(\theta_2) = m_2 R_1^2 \ddot{\theta}_2 + m_2 R_1 R_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - m_2 R_1 R_2 \ddot{\theta}_1^2 \sin(\theta_1-\theta_2)$$