Proof of Euler's Identity

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Abstract

This paper illustrates the proof of Euler's identity.

Contents

1. Euler's Formula and Euler's Identity

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{i\pi} + 1 = 0$$

2. Tayler series and Complex Numbers

2.1. Tayler Series Expansion

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots$$

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \frac{1}{5!}z^5 + \cdots + \frac{1}{n!}z^n$$

2.2. Complex Numbers

$$i^1 = i = \sqrt{-1}$$
$$i^2 = (\sqrt{-1})^2 = -1$$
$$i^3 = (\sqrt{-1})^2 \cdot \sqrt{-1} = -1 \cdot i = -i$$
$$i^4 = (\sqrt{-1})^2 \cdot (\sqrt{-1})^2 = -1 \cdot -1 = 1$$
$$i^5 = i^4 \cdot i^1 = 1 \cdot i = i$$

3. Euler Identity Expansion

$$z = i\theta$$

$$e^z = e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \ldots$$

$$= 1 + i\theta + \frac{(-1)}{2!}\theta^2 + \frac{(-i)}{3!}\theta^3 + \frac{1}{4!}\theta^4 + \frac{i}{5!}\theta^5 + \ldots$$

$$= 1 + i\theta - \frac{1}{2!}\theta^2 - \frac{i}{3!}\theta^3 + \frac{1}{4!}\theta^4 + \frac{i}{5!}\theta^5 + \ldots$$

$$= 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5\right) + \ldots$$

$$= \cos(\theta) + i\sin(\theta)$$

$$e^z = \cos(\theta) + i\sin(\theta)$$

4. Euler Identity Proof

$$e^z = \cos(\theta) + i\sin(\theta)$$

$$\theta = \pi$$

$$e^{i\pi} = \cos(\pi) + i\sin(\pi)$$

$$e^{i\pi} = -1 + i \cdot 0$$

$$e^{i\pi} = -1$$

$$e^{i\pi} + 1 = 0$$

5. Unit Circle