Equation of Motion of a Pendulum

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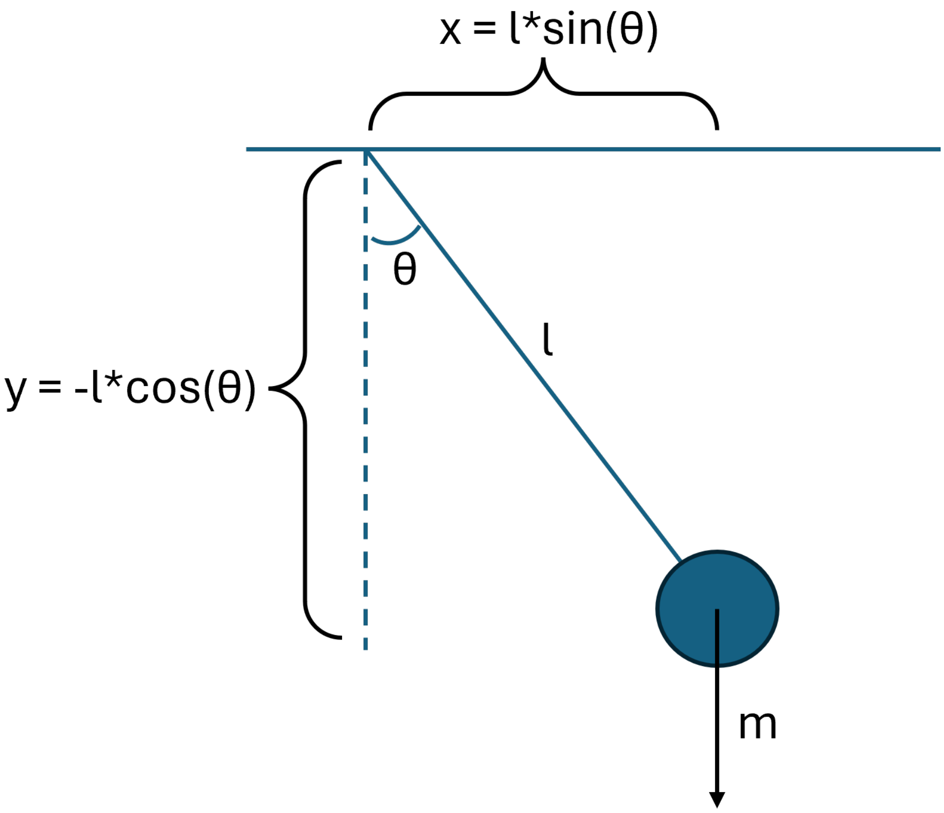
Abstract

This paper describes the mathematics of the equation of motion of a pendulum using the Euler Lagrange equation.

Contents

1. Pendulum

1.1. Free Body Diagram



1.2. Definitions

$$l = \text{length of the pendulum rod}$$
$$m = \text{mass of the pendulum bob}$$
$$g = \text{acceleration due to gravity}$$
$$\theta(t) = \text{angular position}$$
$$\dot{\theta}(t) = \text{angular velocity}$$
$$\ddot{\theta}(t) = \text{angular acceleration}$$

2. Lagrangian Derivation of the Simple Pendulum

2.1. Position Vector

- Length definition

$$l = \text{constant}$$

- Position coordinates

$$x = l\sin(\theta)$$

$$y = -l\cos(\theta)$$

- Position vector

$$\vec{r}_{z,q} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l\sin(\theta) \\ -l\cos(\theta) \\ 0 \end{pmatrix}$$

2.2. Velocity Vector

- Velocity vector

$$\dot{\vec{r}}_{z,q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} l\cos(\theta) \\ l\sin(\theta) \\ 0 \end{pmatrix} \dot{\theta} = \begin{pmatrix} l\cos(\theta)\dot{\theta} \\ l\sin(\theta)\dot{\theta} \\ 0 \end{pmatrix}$$

- Velocity squared

$$\dot{\vec{r}}_{z,q}\dot{\vec{r}}^T_{z,q} = \begin{pmatrix} l\cos(\theta)\dot{\theta} \\ l\sin(\theta)\dot{\theta} \\ 0 \end{pmatrix} \begin{pmatrix} l\cos(\theta)\dot{\theta} & l\sin(\theta)\dot{\theta} & 0 \end{pmatrix}$$

$$\begin{aligned} \dot{\vec{r}}_{z,q} \cdot \dot{\vec{r}}^T_{z,q} &= (l\cos(\theta)\dot{\theta})^2 + (l\sin(\theta)\dot{\theta})^2 \\ &= l^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) \\ &= l^2\dot{\theta}^2 \end{aligned}$$

2.3. Kinetic Energy

- Kinetic energy

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{\vec{r}}_{z,q} \cdot \dot{\vec{r}}^T_{z,q}$$

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

2.4. Potential Energy

- Potential energy

$$U = mgh$$

$$U = mg(-l\cos(\theta)) = -mgl\cos(\theta)$$

2.5. Lagrangian

- Lagrangian equation

$$L = T - U$$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - (-mgl\cos(\theta))$$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos(\theta)$$

2.6. Euler Lagrange

- Euler–Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

- Derivatives

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}(ml^2\dot{\theta}) = ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin(\theta)$$

2.7. Equation of Motion

- Equation of motion

$$ml^2\ddot{\theta} - (-mgl\sin(\theta)) = 0$$
$$ml^2\ddot{\theta} + mgl\sin(\theta) = 0$$
$$l\ddot{\theta} + g\sin(\theta) = 0$$
$$\ddot{\theta} = -\frac{g}{l}\sin(\theta)$$